

# Soluble Models of Strongly Interacting Two-level Ultracold Gases in Tight Waveguides with Coupling to the Quantized Electromagnetic Field

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(Dated: February 1, 2008)

A generalized Fermi-Bose mapping method is used to determine the exact ground states of six models of strongly interacting ultracold gases of two-level atoms in tight waveguides, which are generalizations of the Tonks-Girardeau (TG) gas (1D Bose gas with point hard cores) and fermionic Tonks-Girardeau (FTG) gas (1D spin-aligned Fermi gas with infinitely strong zero-range attractions). Three of these models exhibit a quantum phase transition in the presence of an external magnetic field, associated with a cooperative ground state rearrangement wherein Fermi energy is traded for internal excitation energy. After investigation of these models in the absence of an electromagnetic field, one is generalized to include resonant interactions with a single photon mode, leading to a possible thermal phase transition associated with Dicke superradiance.

PACS numbers: 03.75.-b, 03.75.Mn, 42.50.-p

The rapidly increasing sophistication of experimental techniques for probing ultracold gases has caused a shift of emphasis in theoretical and experimental work in recent years, from effective field approaches to more refined methods capable of dealing with strong correlations. In ultracold gases confined in de Broglie waveguides with transverse trapping so tight that the atomic dynamics is essentially one-dimensional [1], with confinement-induced resonances [1, 2] allowing Feshbach resonance tuning [3] of the effective 1D interactions to very large values, such correlations are greatly enhanced. This has allowed experimental verification [4, 5, 6] of the fermionization of bosonic ultracold vapors in such geometries predicted by the Fermi-Bose (FB) mapping method [7]. Here the FB mapping method used in [7] to solve the TG gas and in [2, 8, 9] to solve the FTG gas is generalized to obtain exact solutions of six models of strongly interacting 1D ultracold gases of 2-level atoms in tight waveguides, with and without resonant coupling to a single mode of the quantized electromagnetic field.

Consider first some 1D gases of ultracold two-level atoms with no coupling to the quantized electromagnetic field. The procedure for obtaining the exact solutions is similar to that used in recent work [10] on two-component mixtures of 1D gases, but with an important difference: In a mixture of dissimilar species there is no symmetry requirement on exchange of dissimilar atoms, whereas in the present case of a single species with two internal levels  $g$  (ground) and  $e$  (excited), the wave function must be symmetric (Bose) or antisymmetric (Fermi) under combined exchange of spatial coordinates and internal states. Starting as in [10] from “model states”  $\Psi_M$  which are ideal Fermi or Bose gases, one can generate the interactions by a mapping  $\Psi_M \rightarrow \Psi = A\Psi_M$  where  $A$  is  $\pm 1$  everywhere and has discontinuities at pair contact which introduce zero-range TG and/or FTG interactions.

Denote spatial coordinates of the  $N$  atoms by  $x_1, \dots, x_N$  and their internal state labels by  $s_1, \dots, s_N$

where each  $s_j$  takes on the values  $g$  or  $e$ . Models I, II, and III start from the model states  $\Psi_{\alpha M}^F$  of an ideal Fermi gas, and models IV, V, and VI from those  $\Psi_{\alpha M}^B$  of an ideal Bose gas:

$$\Psi_{\alpha M}^F = \det_{j,\ell=1}^N \phi_{\nu_j}(x_\ell, s_\ell), \quad \Psi_{\alpha M}^B = \text{alt}_{j,\ell=1}^N \phi_{\nu_j}(x_\ell, s_\ell) \quad (1)$$

where  $\text{alt}$  denotes an alternant (minus signs in the determinant  $\det$  replaced by plus signs). The orbitals  $\phi_\nu(x, s)$  are a complete set of energy eigenstates of a single-particle Hamiltonian  $\hat{H}(x, s) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + v(x, s) + \delta_{sg}\epsilon_g + \delta_{se}\epsilon_e$  appropriate to given boundary conditions and external potential  $v(x, s)$ , satisfying the orthonormality relation  $\sum_{s=g,e} \int dx \phi_\nu^*(x, s) \phi_{\nu'}(x, s) = \delta_{\nu\nu'}$ . The  $\phi_{\nu_j}$  in the Slater determinant  $\Psi_M^F$  are a selection of  $N$  different  $\phi_\nu$ , whereas those in the alternant  $\Psi_M^B$  are such a selection but with multiple occupancy allowed. The terms depending on the internal energy levels  $\epsilon_g$  and  $\epsilon_e$  account for the energy difference  $\epsilon_e - \epsilon_g$  between ground and excited states.  $\Psi_M^F$  and  $\Psi_M^B$  are eigenstates of the interaction-free model Hamiltonian  $\hat{H}_M = \sum_{j=1}^N \hat{H}(x_j, s_j)$  with eigenvalue  $\sum_{j=1}^N \epsilon_{\nu_j}$ , and all selections of the  $\nu_j$  give all such  $N$ -atom energy eigenstates. The Bose ground state is a completely Bose-Einstein condensed state with all  $N$  atoms in the lowest orbital  $\nu = 0$ , i.e.,  $\Psi_{0M}^B = \prod_{j=1}^N \phi_0(x_j, s_j)$ , and the Fermi ground state is a Slater determinant of the lowest  $N$  orbitals, i.e.,  $\Psi_{0M}^F = \det_{\nu,\ell=(0,1)}^{(N-1,N)} \phi_\nu(x_\ell, s_\ell)$ . Assume harmonic trapping with the same longitudinal frequency for both internal ground and excited atoms, i.e.,  $v(x, s) = \frac{1}{2}m\omega^2 x^2$ . Then if  $\epsilon_{eg} = \epsilon_e - \epsilon_g < N\hbar\omega$ , the state with all  $N$  atoms in internal level  $g$  is not the ground state, because the energy can be lowered by exciting the atom at the top of the Fermi sea to internal level  $e$  and moving it to the lowest harmonic oscillator level, signalling an instability of the putative ground state against internally exciting atoms near the top of the Fermi sea and moving them to levels near the bottom, which become doubly occupied

with one g-atom and one e-atom. If  $\epsilon_{eg}$  is a hyperfine splitting, this inequality cannot be satisfied for currently achievable values of  $N$ , so in the absence of an external magnetic field the  $N$ -atom ground states of these models consist of atoms in their ground levels g. However, in the presence of an external magnetic field in the proper direction one has  $\epsilon_{eg} = \epsilon_{\text{HF}} - \epsilon_{\text{Zeeman}}$  where  $\epsilon_{\text{HF}}$  and  $\epsilon_{\text{Zeeman}}$  are the hyperfine and Zeeman splittings. This can be made arbitrarily small by tuning the magnetic field, so a quantum phase transition of this type should be experimentally realizable in a TG gas. It also has intrinsic theoretical interest because it is induced by the strong interatomic interactions and cannot occur in an ideal Fermi gas.

The energy eigenstates including TG and/or FTG interatomic interactions are generated by multiplication of the ideal Fermi and Bose gas states by appropriate mapping functions  $A(x_1, s_1, \dots, x_N, s_N)$ . The original FB mapping solution for the TG gas [7] starts from a model state  $\Psi_F$  which is an ideal spin-aligned Fermi gas Slater determinant, and generates the energy eigenstates  $\Psi_B$  of a system of bosons with TG (point hard core) interactions by multiplying  $\Psi_F$  by a “unit antisymmetric mapping function”

$$A(x_1, \dots, x_N) = \prod_{1 \leq j < \ell \leq N} \text{sgn}(x_j - x_\ell) \quad (2)$$

where the sign function  $\text{sgn}(x)$  is +1 (−1) if  $x > 0$  ( $x < 0$ ). This changes the antisymmetry nodes at  $x_j = x_\ell$  into collision cusps which are the  $c \rightarrow +\infty$  limit of the collision cusps in the Lieb-Liniger solution for a Bose gas with interactions  $c\delta(x_j - x_\ell)$  [11]. The FTG gas is a “mirror image” of the TG gas, consisting of spin-aligned fermions with infinitely-strong zero-range attractions which are a zero-width, infinite depth limit of a square well of depth  $V_0$  and width  $2x_0$ , with the limit taken such that  $V_0 x_0^2 \rightarrow (\pi\hbar^2)/8\mu$  where  $\mu$  is the effective mass of the colliding pair [2, 8, 9]. It causes odd-wave scattering (1D analog of 3D p-wave scattering) with 1D scattering length  $a_{1D} = -\infty$ , with the result that all energy eigenstates  $\Psi_F$  of the FTG gas are obtained from corresponding ideal Bose gas states  $\Psi_B$  (the model states) by the mapping  $\Psi_B \rightarrow \Psi_F = A\Psi_B$ , where  $A$  is exactly the same mapping (2), which now introduces sign-changing discontinuities in the FTG states necessary to reconcile fermionic antisymmetry with a strong interaction in the zero-range limit  $x_0 \rightarrow 0$  [12]. Inside the square well the solution passes smoothly through a zero at  $x_j - x_\ell = 0$ , so the discontinuity is an illusion of the zero-range limit [2, 8, 9, 12].

*Model I:* This is a Bose gas with TG gg and ee interactions and FTG ge interactions. Its energy eigenstates  $\Psi_\alpha^B$  are generated by mapping from the two-level ideal Fermi gas model states  $\Psi_{\alpha M}^F$  of Eqs. (1) according to  $\Psi_\alpha^B = A\Psi_{\alpha M}^F$  where the mapping function  $A$  is the simple one (2) of the original TG gas solution [7].

This generates TG gg and ee interactions, but FTG ge interactions. This is easily seen by a simple example. Suppose that g and e atoms see the same harmonic trap potential  $v(x, s) = \frac{1}{2}m\omega^2 x^2$ . Then the lowest state  $\Psi_{\alpha M}^F$  with 2 atoms in the ground level g and 1 atom in the excited level e is a Slater determinant (1) constructed from the three harmonic oscillator orbitals  $\phi_\nu(x, s)$  where  $\phi_0(x, s) = u_0(x)\delta_{sg}$ ,  $\phi_1(x, s) = u_0(x)H_1(Q)\delta_{sg}$ , and  $\phi_2(x, s) = u_0(x)\delta_{se}$ , where  $Q = x/x_{\text{osc}}$ ,  $x_{\text{osc}} = \sqrt{\hbar/m\omega}$ ,  $u_0(x) = \text{const.}e^{-Q^2/2}$ , and  $H_n$  are Hermite polynomials, as in [13]. Dropping a normalization constant and multiplying by  $A$  of Eq. (2) to obtain  $\Psi^B$ , one finds

$$\begin{aligned} \Psi^B &= u_0(x_1)u_0(x_2)u_0(x_3) \\ &\times [-|x_1 - x_2|\text{sgn}(x_1 - x_3)\text{sgn}(x_2 - x_3)\delta_{s_1g}\delta_{s_2g}\delta_{s_3e} \\ &+ |x_1 - x_3|\text{sgn}(x_1 - x_2)\text{sgn}(x_2 - x_3)\delta_{s_1g}\delta_{s_3g}\delta_{s_2e} \\ &- |x_2 - x_3|\text{sgn}(x_1 - x_2)\text{sgn}(x_1 - x_3)\delta_{s_2g}\delta_{s_3g}\delta_{s_1e}] \cdot \quad (3) \end{aligned}$$

If atoms 1 and 2 are both in level g, then  $\Psi^B$  has a TG  $|x_1 - x_2|$  cusp at  $x_1 = x_2$ , but if atom 1 is in level g and atom 2 in level e or vice versa, then  $\Psi^B$  has a FTG discontinuity at  $x_1 = x_2$ . Similar remarks apply to the atom pairs (1,3) and (2,3). This generalizes to eigenstates  $\Psi_\alpha^B$  with arbitrary numbers of atoms in levels g and e, showing that two atoms both in level g or both in level e have TG cusps at their collision points  $x_j = x_\ell$ , but if one is in level g and the other in level e, there is a FTG discontinuity instead. This model is equivalent to that of Mousavi *et al.* [14], although the equivalence is not immediately apparent since the notation in [14] is very different from that used here, and the interactions in [14] are defined differently, as TG in the three triplet gg, ee, and ge channels and FTG in the singlet ge channel [15]. To see the equivalence note that (3) can be rewritten in terms of triplet and singlet ge channels as [16]

$$\begin{aligned} \Psi^B &= \frac{1}{4}u_0(x_1)u_0(x_2)u_0(x_3)\{ |x_1 - x_2| \\ &\times \text{sgn}(x_1 - x_3)\text{sgn}(x_2 - x_3)\delta_{s_3g}(\delta_{s_1g}\delta_{s_2e} + \delta_{s_2g}\delta_{s_1e}) \\ &- |x_1 - x_3|\text{sgn}(x_1 - x_2)\text{sgn}(x_2 - x_3) \\ &\times \delta_{s_2g}(\delta_{s_1g}\delta_{s_3e} + \delta_{s_3g}\delta_{s_1e}) \\ &+ |x_2 - x_3|\text{sgn}(x_1 - x_2)\text{sgn}(x_1 - x_3) \\ &\times \delta_{s_1g}(\delta_{s_2g}\delta_{s_3e} + \delta_{s_2g}\delta_{s_3e}) \\ &+ \text{sgn}(x_1 - x_2)\text{sgn}(x_1 - x_3)\text{sgn}(x_2 - x_3) \\ &\times [(x_1 - 2x_3 + x_2)\delta_{s_3g}(\delta_{s_1g}\delta_{s_2e} - \delta_{s_2g}\delta_{s_1e}) \\ &+ (x_3 - 2x_2 + x_1)\delta_{s_2g}(\delta_{s_3g}\delta_{s_1e} - \delta_{s_1g}\delta_{s_3e}) \\ &+ (x_2 - 2x_1 + x_3)\delta_{s_1g}(\delta_{s_2g}\delta_{s_3e} - \delta_{s_3g}\delta_{s_2e})] \} \cdot \quad (4) \end{aligned}$$

It follows that the interaction in the triplet ge channels is of TG form [17] whereas that in the singlet ge channels is of FTG form, but the equal expression (3) is much simpler due to cancellations in (4), does not require separation into singlet and triplet channels, yet exhibits the TG interaction in the triplet gg channel, which is not

evident in the form (4). The model represents a two-level hybrid TG-FTG gas with infinitely strong repulsive gg and ee TG interactions and infinitely strong *attractive* ge FTG interactions, although it was called simply a two-level TG gas in [14]. Its experimental realization would be difficult since generation of the FTG interaction would require a p-wave ge resonance, and one would have to simultaneously create TG gg and ee interactions, although the latter could perhaps be done by the methods of [4, 5, 6].

*Model II:* This is a Bose gas with TG gg and ee interactions but no ge interactions. It starts from the same ideal Fermi gas model states  $\Psi_{\alpha M}^F$  of Eqs. (1) and generates the states  $\Psi_{\alpha}^B$  with TG gg and ee interactions by a mapping  $\Psi_{\alpha}^B = A\Psi_{\alpha M}^F$ , but the mapping is now a more complicated one depending on both spatial and internal variables, which is everywhere  $\pm 1$  and antisymmetric under exchanges  $(x_j, s_j) \leftrightarrow (x_{\ell}, s_{\ell})$ :

$$A(x_1, s_1; \dots; x_N, s_N) = \prod_{1 \leq j < \ell \leq N} \alpha(x_j, s_j; x_{\ell}, s_{\ell})$$

$$\alpha(x_j, s_j; x_{\ell}, s_{\ell}) = \delta_{s_j, s_{\ell}} \text{sgn}(x_j - x_{\ell}) + \delta_{s_j, g} \delta_{s_{\ell}, e} - \delta_{s_j, e} \delta_{s_{\ell}, g} . \quad (5)$$

This is the same mapping used previously in a model of a TG Mach-Zehnder interferometer, where the state variable  $s$  distinguished between the two interferometer arms [18]. It generates a TG interaction in the gg and ee channels  $\delta_{s_j, s_{\ell}}$ , but no ge interaction. It should be easier to realize experimentally than model I.

*Model III:* This is a *Fermi* gas with FTG ge interactions but no gg or ee interactions. It again starts with the ideal Fermi gas model states  $\Psi_{\alpha M}^F$  of Eqs. (1), but uses a different mapping  $A$ , which is everywhere  $\pm 1$  and is now *symmetric* under exchanges  $(x_j, s_j) \leftrightarrow (x_{\ell}, s_{\ell})$ , to generate the states  $\Psi_{\alpha}^F$  with FTG interactions:

$$A(x_1, s_1; \dots; x_N, s_N) = \prod_{1 \leq j < \ell \leq N} \alpha(x_j, s_j; x_{\ell}, s_{\ell})$$

$$\alpha(x_j, s_j; x_{\ell}, s_{\ell}) = \delta_{s_j, s_{\ell}} + (\delta_{s_j, g} \delta_{s_{\ell}, e} - \delta_{s_j, e} \delta_{s_{\ell}, g}) \text{sgn}(x_j - x_{\ell}) . \quad (6)$$

It generates no interaction in the gg and ee channels  $\delta_{s_j, s_{\ell}}$ , but FTG ge interactions.

*Models IV, V, and VI:* These all start from the ideal two-level Bose gas model states  $\Psi_{\alpha M}^B$  of Eqs. (1). Model IV uses the simple mapping (2) and has FTG gg, ee, and ge interactions, model V uses the mapping (5) and has FTG gg and ee interactions, and model VI uses the mapping (6) and has FTG ge interactions. Model VI should be the easiest to realize since a p-wave Feshbach resonance is required only in the ge channel.

*Interaction with a coherent photon mode:* Some thirty years ago I suggested the possibility of a cooperative coupling between Bose-Einstein condensation (BEC) in liquid  $^4\text{He}$  and BEC of van der Waals virtual photons into

a superradiant mode, and constructed a simple model exhibiting such a quantum phase transition [19]. Such equilibrium superradiance has never been observed (nor, I believe, searched for) in superfluid  $^4\text{He}$ , but in recent years coupling of matter waves and a superradiant photon mode have been observed in ultracold Bose gases [20, 21], and it has been pointed out that essentially the same phenomena can also occur in ultracold Fermi gases [22, 23]. In fact, it had been shown long ago by Hepp and Lieb [24] that BEC of the atoms is not necessary for co-operative coupling of internal and translational degrees of freedom in a multilevel gas via a superradiant photon mode. Motivated by these results, model I above will be generalized by adding resonant coupling to a single mode of the quantized electromagnetic field [25].

Start with the atom-field interaction Hamiltonian in the electric field gauge and rotating wave approximation:  $\hat{H}_{\text{atom-field}} = i \sum_{j=1}^N \sum_{k\lambda} \sqrt{\frac{\hbar c}{L}} (\mathbf{d}_{eg} \cdot \mathbf{e}_{k\lambda} \hat{S}_j^+ \hat{b}_{k\lambda} e^{ikx_j} - \text{H.c.})$ . Here  $\hat{b}_{k\lambda}$  and  $\hat{b}_{k\lambda}^\dagger$  are annihilation and creation operators for photons with wave vector  $k$  and polarization  $\lambda$ ,  $\hat{S}_j^+$  and  $\hat{S}_j^- = (\hat{S}_j^+)^\dagger$  are raising and lowering operators for the internal levels in the usual spin  $\frac{1}{2}$  representation,  $\mathbf{d}_{eg}$  is the transition dipole moment from level  $g$  to level  $e$ ,  $\mathbf{e}_{k\lambda}$  are unit polarization vectors, and the allowed  $k$  are integer multiples of  $2\pi/L$  where  $L$  is the length of a 1D microwave cavity resonant with the hyperfine transition  $g \leftrightarrow e$ . Retaining coupling only to a single mode of wavelength  $\lambda = \frac{\hbar c}{\epsilon_{eg}}$  resonant with the hyperfine transition, one obtains a Hamiltonian generalizing that of models I-III:  $\hat{H} = \hbar c q \hat{N}_q + \sum_{j=1}^N [-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_j^2} + v(x_j) + \epsilon_{eg} \hat{S}_j^z + i\gamma L^{-1/2} (\hat{S}_j^+ \hat{b}_q e^{iqx_j} - \text{H.c.})]$  where  $q = \frac{\epsilon_{eg}}{\hbar c}$ ,  $\hat{N}_q = \hat{b}_q^\dagger \hat{b}_q$ ,  $\gamma = \sqrt{\hbar c q} \mathbf{d}_{eg} \cdot \mathbf{e}_q$ , a constant term  $\frac{N}{2} (\hbar c q + \epsilon_g + \epsilon_e)$  has been dropped, and the external potential is taken here to be that of a longitudinal harmonic trap,  $v(x) = \frac{1}{2} m \omega^2 x^2$ . To obtain the ground state with both coupling to the electromagnetic field and TG and/or FTG interatomic interactions, one can first let the above Hamiltonian act on the space of ideal Fermi gas model states  $\Psi_{\alpha M}^F$ , find the ground state  $\Psi_{0M}^F$  in that space, and then find the interacting ground state  $\Psi_0^B$  by application of the mapping (2).

The length of typical cigar-shaped ultracold gas traps of high aspect ratio is many orders of magnitude smaller than relevant microwave wavelengths, so it is an excellent approximation to replace  $e^{iqx_j}$  by unity (dipole approximation for the whole  $N$ -atom system), leading to a zero-order Hamiltonian  $\hat{H}_0 = \hat{H}_{\text{trans}} + \hat{H}_{\text{Dicke}}$  where the translational Hamiltonian is  $\hat{H}_{\text{trans}} = \sum_{j=1}^N [-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_j^2} + v(x_j)]$  and the remainder is a resonant Dicke model,  $\hat{H}_{\text{Dicke}} = \epsilon_{eg} (\hat{N}_q + \hat{S}^z) + i\gamma L^{-1/2} \sum_j (\hat{S}_j^+ \hat{b}_q - \text{H.c.})]$  where  $\hat{S}^z = \sum_j \hat{S}_j^z$  and the resonance condition  $\hbar c q = \epsilon_{eg}$  has been inserted. There is strong statistical field-atom coupling since the eigenstates of  $\hat{H}_{\text{trans}}$  are those of an ideal Fermi

gas which must be antisymmetric under combined space-spin exchange  $(x_j, s_j) \leftrightarrow (x_\ell, s_\ell)$ .

The ground state of  $\hat{H}_{\text{Dicke}}$  in the thermodynamic limit is the direct product of a field-independent state and a Glauber coherent state for the electromagnetic field [26, 27]. This amounts to replacement of the photon annihilation and creation operators by c-numbers  $\beta_q$  and  $\beta_q^*$ . The ground state energy is invariant under a gauge transformation of the first kind  $\beta_q \rightarrow \beta_q e^{i\theta}$  with  $\theta$  arbitrary; a convenient choice here is  $\beta_q = -i\sqrt{N}f_q$  where  $f_q$  is real, nonnegative, and independent of  $N$ . The corresponding reduced Hamiltonian is  $\hat{H}_0 = \epsilon_{eg}Nf_q^2 + \sum_{j=1}^N [-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_j^2} + v(x_j) + \epsilon_{eg}\hat{S}_j^z + \gamma\sqrt{\rho_q}f_q(\hat{S}_j^+ + \hat{S}_j^-)]$  where  $\rho_q = N/L$ . The single-atom energies  $\epsilon_n^\pm$  and eigenstates  $\phi_n^\pm$  can be found by a  $2 \times 2$  matrix diagonalization for each  $j$ , with the results  $\epsilon_n^\pm = n\hbar\omega \pm \frac{1}{2}\sqrt{\epsilon_{eg}^2 + 4\gamma^2\rho_q f_q}$  and  $\phi_n^\pm(x, s) = u_n(x)w_n^\pm(s)$  where  $n = j - 1 = 0, 1, 2, \dots$ ,  $u_n(x)$  are the harmonic oscillator eigenfunctions, and  $w_n^\pm(s = \frac{1}{2})/w_n^\pm(s = -\frac{1}{2}) = (\epsilon_{eg} \pm \sqrt{\epsilon_{eg}^2 + 4\gamma^2\rho_q f_q})/2\gamma\sqrt{\rho_q f_q}$ . The quantum phase transition previously described in the presence of an external magnetic field  $B$  will still occur, but now at a higher value of  $B$  since  $\epsilon_n^+ - \epsilon_n^- > \epsilon_{eg}$ .

The above assumes that the trapped ultracold gas is bathed in a constant microwave field which is supplied and controlled externally. If it is instead contained in a perfectly reflecting cavity with no external microwave source, one can investigate the possibility of a Dicke thermal phase transition to a state with a self-generated superradiant microwave field. Minimizing the grand canonical free energy  $-\beta^{-1} \ln \text{Tr } e^{-\beta(\hat{H}_0 - \mu\hat{N})}$  of the ideal Fermi gas of models I-III with respect to  $f_q$  [26, 27] yields the following condition for determining  $f_q$ :  $2N\epsilon_{eg}f_q = \frac{\gamma^2\rho_q}{\sqrt{\epsilon_{eg}^2 + 4\gamma^2\rho_q f_q}} \sum_{n=0}^{\infty} (g_n^- - g_n^+)$  where  $g_n^\pm = [1 + e^{\beta(\epsilon_n^\pm - \mu)}]^{-1}$  and  $\mu = \epsilon_F$  is determined from  $\sum_{n=0}^{\infty} (g_n^+ + g_n^-) = N$ . Even for the much simpler case of fixed atoms treated classically (no Fermi sea), solution for the superradiant phase transition is nontrivial, so I shall not proceed further now. However, by comparison with that case [26, 27] it is reasonable to conjecture that there is no phase transition for small values of the ratio  $\gamma\sqrt{\rho_q}/\epsilon_{eg}$  but a thermal transition to a superradiant phase for large values of this ratio. Recalling that  $\epsilon_{eg}$  can be made arbitrarily small by tuning an external magnetic field  $B$ , it seems likely that this model will exhibit a superradiant phase transition for sufficiently large  $B$ , which may be coupled to the previously described cooperative rearrangement of the translational ground state. Note that arguments against existence of a superradiant transition in real atomic systems [28] do not apply here, since  $\epsilon_{eg}$  includes a negative Zeeman shift due to  $B$ .

I am grateful to Gonzalo Muga for detailed discussions of [14], to Ewan Wright for helpful suggestions, and to the U.S. Office of Naval Research for partial support of

my research.

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- [17] Note that at  $x_1 = x_2$  the product of sgn factors in the first term of (4) is equal to 1, so there are no sgn discontinuities in the 1, 2 triplet channel. Similar remarks apply to the 1, 3 and 2, 3 triplet channels.
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